Collectivity, phase transitions, and exceptional points in open quantum systems

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Phase transitions in open quantum systems, which are associated with the formation of collective states of a large width and of trapped states with rather small widths, are related to exceptional points of the Hamiltonian. Exceptional points are the singularities of the spectrum and eigenfunctions, when they are considered as functions of a coupling parameter. In the present paper this parameter is the coupling strength to the continuum. It is shown that the positions of the exceptional points (their accumulation point in the thermodynamical limit) depend on the particular type and energy dependence of the coupling to the continuum in the same way as the transition point of the corresponding phase transition. [S1063-651X(98)07509-6]

I. INTRODUCTION

Recently, mechanisms of restructuring of quantum systems have been discussed with renewed interest [1–7]. Conditions for the formation of collective states on the one hand and of quantum chaos on the other are being studied. Among the oldest examples of the former is the schematic model explaining in simple terms the origin of giant dipole resonances in nuclei [8]. The Hamiltonian is of the form $H = H_0 + H_1 = H_0 + DD^T$ where H_0 is the Hamiltonian of the unperturbed system and DD^T describes the factorized dipole-dipole residual interaction. The rank of H_0 is N where N is the number of unperturbed states considered in a certain energy interval, while the rank of DD^T is 1. If the modulus of the average matrix elements of the vector D is sufficiently large ($\bar{D} \gg d_0$, where $\bar{D}$ is the average matrix element of D and d_0 is the mean level distance of the eigenvalues of H_0), one of the eigenstates of H is shifted considerably in energy. Since its eigenvector has contributions from almost all basis states, it is a collective state. This type of collectivity is called *internal* collectivity [4].

Besides the internal collectivity, there exists the so-called external collectivity of resonance states [4]. It appears at high level density since the discrete states are in general embedded in the continuum and coupled to each other via the continuum. The Hamiltonian of the open quantum system is, to a good approximation, given by $H_0 - iVV^T$ where VV^T contains the coupling matrix elements V_j^c between the discrete states j and the decay channels c. The rank of VV^T is equal to the number K of open decay channels. It is usually much smaller than the rank N of H_0. If the matrix elements of VV^T are sufficiently large ($\bar{V} \gg d_0$), K collective states are formed. They are distinguished from the $N-K$ noncollective states by their large widths or short lifetimes [1–7]. The wave functions of these K collective resonance states are again characterized by a large number of components of the basis states.

There is a basic difference between internal and external collectivity. Internal collectivity deals with a self-adjoint Hamiltonian, where the coupling parameter is real, while external collectivity is described by a purely imaginary coupling strength. The particular type of collectivity is therefore expressed by a real spectrum in the first case, whereas in the second case the dissipative character of the open system is reflected by imaginary parts of the eigenenergies, the physical widths. Formally, the formation of a collective state is, however, the same in both cases. It is expressed by the mixing of the eigenfunctions of the full Hamiltonian with respect to the eigenfunctions of H_0.

The case $\text{rank}(H_0) = \text{rank}(H_1)$ is not necessarily connected with the appearance of collective states. It has been widely discussed in the literature [9]. Conditions have been given for the occurrence of chaotic behavior and of some sort of phase transitions [10]. The mechanism of such restructuring and possible chaotic behavior in the transitional region ($\Lambda \sim \Lambda_c$) have been explained in terms of the exceptional points (EPs) of the problem $H_0 + \Lambda H_1$.

The EPs are the only singularities in the complex Λ plane of the eigenvalues $E_k(\Lambda)$. Their positions are fixed by the choice of H_0 and H_1 only. As a consequence, the distribution of the EPs is characteristic for any particular Hamiltonian of the form $H_0 + \Lambda H_1$, once H_0 and H_1 are given.

It is therefore natural to discuss the EPs as they determine important properties of the spectrum such as the statistical properties, the type and locations of avoided level crossings, the softness of the spectrum, and the ranges of Λ values, where special features of the spectrum occur. In particular, the distribution and frequency of the EPs give indications about transitional regions and possible occurrence of chaotic behavior.

The studies of the EPs have, in the papers quoted, been dealing with closed systems, where the influence of decay channels in the continuum has not been considered, that is, H is self-adjoint and the eigenstates are discrete. As stated above, we are often faced with dissipative systems, where the coupling to the continuum and decay channels is expressed by complex eigenenergies which give rise to finite lifetimes of resonance states. The complex eigenvalues $\mathcal{E}_k = E_k - i/2\Gamma_k$ are interpreted as resonance states at the energies E_k with the decay widths Γ_k which are inverse lifetimes.

While there is a great mathematical similarity in the treat-

ment of the respective self-adjoint and dissipative Hamiltonians, the physical findings deserve particular attention. Of special interest will be the circumstances under which the formation of collective states can be understood as a phase transition. We generalize the results obtained in [10] to open quantum mechanical systems and expand the investigations of [7] by taking into account the effects and properties of the EPs. In all cases, we will restrict ourselves to the one-channel case in which the phase transition as well as the collectivity of one of the states are well pronounced.

Generally, a phase transition is a substantial restructuring of the system taking place at a finite (critical) value Λ_c of a certain control parameter Λ. This restructuring is a collective phenomenon, ranging over all scales inherent in the system. The nature of the reorganization process is characterized by the behavior of an order parameter being a function of Λ. In our case of an open quantum system the order parameter is Γ_0/N where Γ_0 is the width of the collective mode (in the one-channel case) and N is the total number of states [7]. Its first derivative with respect to Λ shows a finite discontinuity at the critical value Λ_c corresponding to a second-order phase transition. All characteristic features of the phase transition emerge already at finite values of N despite the fact that the strict thermodynamical definition is possible only for $N \rightarrow \infty$. Precise conditions for its occurrence are derived in [7].

The phase transition is connected with the appearance of a state whose external collectivity is of a global nature. Its wave function carries contributions from *all* eigenstates of H_0. This global collectivity must be distinguished from the local (external) collectivity which appears when the resonance states are coupled strongly to the continuum but the conditions for a phase transition are not fulfilled. The wave function of this collective state carries contributions only from a restricted number of basis states, whose unperturbed energies are overlapped by the width of the collective state [7].

In Sec. II the basics of the EPs and specific features relating to open quantum systems are reviewed. A simple example using two resonances coupled to one decay channel illustrates the connection between the main results known from the study of open quantum mechanical systems [3] and the aspect of the EPs. In Sec. III the restructuring of the system is put into context with several systems of a different nature. As a beginning, the simplest case of a picket-fence model is considered. Characteristic patterns of the EPs relating to phase transitions are established. Next, the influence of random perturbations within the system is investigated. Finally we discuss the effect of a more general level density distribution of the unperturbed levels and that of the coupling vector V. A short summary and outlook is given in Sec. IV.

II. EXCEPTIONAL POINTS AND FEATURES OF OPEN QUANTUM SYSTEMS

The $N \times N$ matrix problem of the form $H_0 + \Lambda H_1$ has eigenvalues $E_k(\Lambda)$, $k = 1, \ldots, N$ which are obtained by the secular equation

$$\det(H_0 + \Lambda H_1 - E) = 0. \tag{1}$$

We here assume that H_0 and H_1 are real and symmetric. The EPs [9] are characterized by the coalescence of any two pairs

of eigenvalues. They are fixed by a particular choice of H_0 and H_1. If we exclude a genuine degeneracy of eigenvalues for real values Λ, such coalescence will happen only for complex values of the coupling parameter and hence for complex eigenvalues. Accordingly, the EPs are determined by the simultaneous solutions of Eq. (1) and of

$$\frac{d}{dE}\det(H_0 + \Lambda H_1 - E) = 0, \tag{2}$$

where the second equation ensures that two eigenvalues coincide. Equations (1) and (2) are polynomials in E and Λ of order N and $N-1$, respectively. They can be combined into a single polynomial in Λ of order $N(N-1)$, the resultant of Eq. (1), by eliminating the variable E. The resultant has real coefficients, hence the EPs occur in $N(N-1)/2$ complex conjugate pairs. They are the only singularities which the eigenvalues $E_k(\Lambda)$ can have as functions of Λ. In fact, they are the square root branch points of one analytic function which has N Riemann sheets, where the values on each sheet are just the real eigenvalues $E_k(\Lambda)$ for real values Λ. All N eigenvalues are therefore analytically connected with each other via the square root branch points in the complex Λ plane.

The behavior of the eigenfunctions when continued to the EPs deserves special interest. Since they are lying in the complex Λ plane, the operator $H_0 + \Lambda H_1$ is no longer self-adjoint at $\Lambda = \Lambda_{EP}$ with Λ_{EP} denoting a (complex) EP. Therefore we cannot expect two linearly independent eigenvectors even though two (complex) eigenvalues E_i and E_k coalesce. The eigenbasis of $H_0 + \Lambda_{EP}H_1$ is no longer orthogonal as is the case for self-adjoint Hamiltonians. In fact in contrast to the self-adjoint case, where a twofold degeneracy always implies a two-dimensional eigenspace, an EP is characterized by the fact, that the rank of the associated matrix $H_0 + \Lambda_{EP}H_1$ drops by one. In other words, there is no two-dimensional subspace associated with the coalescence of two eigenvalues. We rather encounter a confluence of the two eigenvectors $\psi_l(\Lambda)$ and $\psi_k(\Lambda)$ at $\Lambda = \Lambda_{EP}$. Moreover, for a general complex value Λ not coinciding with an EP, we may choose a biorthogonal system such that

$$\langle \tilde{\psi}_l(\Lambda) | \psi_k(\Lambda) \rangle = \delta_{lk}, \tag{3}$$

where $\tilde{\psi}_i(\Lambda)$ and $\psi_i(\Lambda)$ are the left and the right eigenvectors, respectively. We note that the transformation O diagonalizing a symmetric complex Hamiltonian H is complex orthogonal ($OO^T = O^TO = 1$) but not unitary. Therefore the scalar product of the two left or right eigenvectors obeys $\langle \tilde{\psi}_k(\Lambda) | \tilde{\psi}_k(\Lambda) \rangle = \langle \psi_k(\Lambda) | \psi_k(\Lambda) \rangle \geq 1$ for any k. A problem occurs when the normalization condition given by Eq. (3) is considered for $\Lambda \rightarrow \Lambda_{EP}$: with $E_l(\Lambda_{EP}) = E_k(\Lambda_{EP})$ it is $\psi_l(\Lambda_{EP}) = \psi_k(\Lambda_{EP})$. This means, in view of Eq. (3), that $\psi_k(\Lambda_{EP})$ cannot be normalized at $\Lambda = \Lambda_{EP}$, since now the orthogonality conflicts with the normalization requirement. Usually, Eq. (3) is globally enforced; as a consequence, the two states $\psi_l(\Lambda)$ and $\psi_k(\Lambda)$ not only coincide for $\Lambda \rightarrow \Lambda_{EP}$ but they blow up, that is $\lim_{\Lambda \rightarrow \Lambda_{EP}} \langle \psi_k(\Lambda) | \psi_k(\Lambda) \rangle \rightarrow \infty$ [3].

The physical significance of the EPs lies in their relation to avoided level crossing. In particular, in a number of ex-

amples it has been demonstrated that in the region of the real values of Λ, where a high density of EPs occurs, the typical statistical characteristics of the spectrum ascribed to quantum chaos prevail [9,11].

Turning to open quantum systems

$$H = H_0 - i\Lambda VV^\dagger \qquad (4)$$

we are faced with a Hamiltonian which has in general complex eigenvalues $\mathcal{E}_k = E_k - i/2\Gamma_k$ as the dimensionless strength parameter Λ is complex,

$$\Lambda = \lambda e^{i\varphi}. \qquad (5)$$

[The factor i in front of the coupling term Λ in Eq. (4) is used traditionally in this context.] The question then arises, whether and under which conditions two (or more) eigenvalues coincide in the complex energy plane and how such a crossing depends on the coupling strength Λ. The answer to these questions depends on the manner by which the EPs are fixed by the operators H_0 and V.

For illustration let us consider the simple example of two resonances, which are coupled to one open decay channel. The Hamiltonian matrix for this system can be written in the eigenbasis of H_0 as

$$H = \begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix} - i\Lambda \begin{pmatrix} \cos^2\omega & \cos\omega\sin\omega \\ \cos\omega\sin\omega & \sin^2\omega \end{pmatrix}$$
$$\equiv H_0 - i\Lambda\, VV^\dagger, \qquad (6)$$

where we use $H_1 = VV^\dagger$. The relative coupling strength of the two resonance states to the continuum is determined by the angle ω of the vector $V = (\cos\omega, \sin\omega)$. The simplicity of the model provides an analytic expression for the two EPs, viz.,

$$\Lambda_{EP} = i(\epsilon_2 - \epsilon_1)e^{\pm 2i\omega}, \qquad (7)$$

which are the zeros of the square root in the expression for the eigenenergies, which read

$$\mathcal{E}_{1,2} = \frac{\epsilon_1 + \epsilon_2 - i\Lambda}{2}$$
$$\pm \frac{1}{2}\sqrt{(\epsilon_1 - \epsilon_2)^2 - 2i\Lambda(\epsilon_1 - \epsilon_2)\cos 2\omega + (i\Lambda)^2}. \qquad (8)$$

The example nicely demonstrates that the EPs depend *only* on H_0 and H_1. Under variation of the angle ω the EPs are moving on a circle with radius $\epsilon_2 - \epsilon_1$ in the complex Λ plane. What determines the EPs are the energies of the unperturbed states (that is, H_0) and their relative coupling (that is, H_1), and *not* the phase φ of the coupling $\Lambda = \lambda e^{i\varphi}$.

Yet, the complex eigenvalues $\mathcal{E}_{1,2}$ obviously *do* depend on Λ, that is, on its modulus λ and on the angle φ. Variation of λ invokes trajectories in the complex energy plane. The con-

tours of the trajectories depend therefore on both the phase φ and the relative coupling controlled by ω. For $\omega = 45°$ and $\varphi = \varphi_{EP} = 0°$ the two trajectories will cross one EP at $\lambda = \lambda_{EP} = \epsilon_2 - \epsilon_1$. At this critical value of the coupling strength the two eigenvalues coalesce. The dimension of the eigenspace of the Hamiltonian is reduced to one including the consequences for the eigenvectors mentioned above. We stress, however, that the crossing of trajectories should not be confused with the familiar degeneracy for self-adjoint Hamiltonians.

For $\varphi \neq \varphi_{EP}$ the two trajectories repel each other within a certain *finite* distance in the complex plane [3]. This is a generalization of avoided level crossing for real eigenvalues of a Hermitian Hamilton operator. Only if φ is properly tuned to the value of ω can the two energy trajectories genuinely cross each other. A nice illustration of this simple example for different values of ω and φ can be found in [6].

What becomes obvious in the 2×2 matrix model can be generalized to an arbitrary $N \times N$ situation, i.e., N resonance states coupled to one common decay channel. With increasing λ one eigenvalue trajectory always drifts further into the complex plane, while the others are bending back towards the real axis after they have repelled with the collective state whose width always increases. This happens irrespective of φ. Physically it means that one of the resonance states takes almost all of the transition strength by trapping the others which then become long-lived [3,6].

This is understood by the rank one of the coupling matrix. At large values of λ the first part H_0 of the Hamiltonian is a small perturbation. Therefore the total matrix is essentially turned into an operator of rank 1. In other words, there is only one nonzero eigenvalue, and the widths of the zero eigenvalues have to vanish. In general, with N resonance states and $K < N$ open decay channels, there appear K fast decaying states and $N - K$ states which are virtually stable. Various examples of the many-channel–many-resonance case can be found in [1–7].

The low rank of H_1 has a drastic effect on the total number of EPs. If K is the rank of H_1, the number of EPs is $K(2N - K - 1)$. The important point is the linear behavior in N; only when K attains the order of magnitude of N is the quadratic behavior retrieved. This finding is significant in that it indicates that chaotic behavior cannot be *generated* by a low rank of H_1, there is simply an insufficient number of avoided level crossings. This is in line with an analysis of the level statistics of a Poisson ensemble coupled to a continuum by a Gaussian coupling vector [2,5], where, for large coupling strength, chaotic features of the system are the more pronounced the larger the number K of decay channels.

III. RESULTS

We turn our attention to the critical region, where a restructuring of the system occurs. We address the relationship between the potential occurrence of a phase transition which is associated with the formation of a globally collective state, and the distribution of the EPs. Since the essential properties of the Hamiltonian must be reflected in the distribution of the EPs, we expect this relationship to exist.

A. Simple example: Picket-fence model

We consider the simple model with

$$
H_0 = \begin{pmatrix} -\dfrac{N-1}{2} & 0 & \cdots & 0 \\[2mm] 0 & -\dfrac{N-3}{2} & \cdots & 0 \\[2mm] \vdots & & \ddots & \vdots \\[2mm] 0 & \cdots & 0 & \dfrac{N-1}{2} \end{pmatrix} \tag{9}
$$

and the $N \times N$ matrix of rank one

$$
VV^\dagger = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix}. \tag{10}
$$

The Hamiltonian describes a picket-fence spectrum, where all states are coupled equally to one decay channel. Such a system shows a phase transition at $\Lambda = 1/\pi$ [7]. Results for finite N can be obtained easily by numerical means. In the limit $N \to \infty$ the zeros of $\det(H_0 - i\Lambda VV^\dagger - \mathcal{E})$ and $\sin(\pi\mathcal{E}) - i\Lambda \cos(\pi\mathcal{E})$ coincide [7]. One finds an accumulation point of the EPs, which emerges in the limit and is found from the zeros of the resultant, which is obtained by eliminating the variable $\mathcal{E}$ from the set

$$
\sin(\pi\mathcal{E}) + i\pi\Lambda \cos(\pi\mathcal{E}) = 0, \tag{11}
$$

$$
\cos(\pi\mathcal{E}) - i\pi\Lambda \sin(\pi\mathcal{E}) = 0. \tag{12}
$$

These two equations are equivalent to Eqs. (1) and (2). Their simultaneous solutions, the zeros of the resultant (denoted by Rsl), are given by

$$
\mathrm{Rsl}(\lambda) = \sqrt{1 + (i\pi\Lambda)^2}. \tag{13}
$$

Obviously, this is no longer a polynomial. The important point of our finding is that *all* roots which occur in the complex Λ plane for finite N converge to $\Lambda = \lambda = \pm 1/\pi$ in the limit $N \to \infty$. Negative values of λ would lead to negative decay widths which have no physical meaning. In the following, we restrict ourselves to positive λ and $0° \leq \varphi \leq 90°$, since all relations are symmetric with respect to the replacement $\Lambda \to -\Lambda$ and $\Lambda \to \Lambda^*$. Note also that the limit point, being a point of accumulation, is no longer a square root branch point for the energy spectrum but rather a logarithmic branch point. In fact, the analytic behavior of the (infinitely many) energy levels $\mathcal{E}_k(\Lambda)$ in the vicinity of $\Lambda = 1/\pi$ is found by solving the secular equation explicitly for $\mathcal{E}$. The expression reads

$$
\mathcal{E}(\Lambda) = \frac{1}{\pi}\arctan(-i\Lambda\pi) = \frac{i}{2\pi}\ln\frac{1-\Lambda\pi}{1+\Lambda\pi}, \tag{14}
$$

which clearly reveals the logarithmic branch points at $\Lambda = 1/\pi$. Moreover, for $\lambda\pi < 1$ we read off

$$
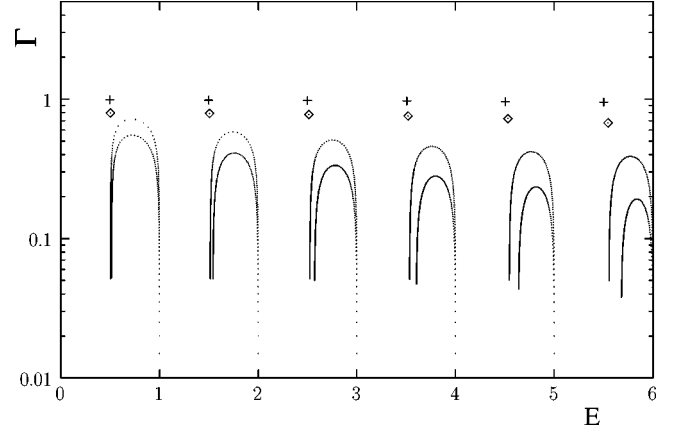\mathcal{E}_k = k - i\Lambda + O(\Lambda^2) \tag{15}
$$

FIG. 1. The trajectories of the eigenvalues for $N = 15$ and $N = 43$ for increasing $\lambda \in [0.001, 2]$ in steps of 0.001. The crossing points are marked by diamonds ($N = 15$) and crosses $N = 43$). E and Γ are given in units of the distance of the unperturbed energies ϵ_k of the picket fence.

while for $\lambda\pi > 1$

$$
\mathcal{E}_k = k + \frac{1}{2} + \frac{i}{2\pi}\ln\left|\frac{1-\Lambda\pi}{1+\Lambda\pi}\right| = k + \frac{1}{2} - \frac{i}{\pi^2\Lambda} + O(\Lambda^{-2}), \tag{16}
$$

with k integer. In both cases, the remaining terms, denoted by O, are purely imaginary if $\varphi = 0°$. These results were obtained in [7].

For finite N the resultant relating to the Hamiltonian $H_0 - i\Lambda VV^\dagger$ becomes a polynomial of order $N - 1$ in Λ^2. The complex roots which are the EPs are therefore not only symmetric with respect to the real axis but a solution Λ_{EP} implies also the solution $-\Lambda_{\mathrm{EP}}$. As a consequence, a solution has to occur on the real Λ axis for N even. This is a nongeneric feature of the present model. For odd values of N and $\varphi = 0°$, no eigenvalue is crossing an EP under variation of λ. We use such cases in our numerical demonstrations.

To illustrate how the $\mathcal{E}$ undergo level repulsions and how the repulsions are related to the crossing points $\mathcal{E}(\Lambda_{\mathrm{EP}})$, we display in Fig. 1 the crossing points and the eigenvalue trajectories for different values of N and $\varphi = 0°$. Since the spectrum is symmetric with respect to positive and negative energies, only the positive part near to the center is drawn. For $\Lambda = 0$ all trajectories begin at the unperturbed energies ϵ_k, move into the complex plane and then turn back ($\epsilon_k \neq 0$) towards the real axis again. For large N the corresponding values are given by Eq. (16) (which is valid for $N \to \infty$). With increasing N the imaginary parts of the turning and the crossing points increase, while they move nearer to each other. The trajectory of the collective state ($\epsilon_k = 0$) is moving on the imaginary axis towards larger imaginary values implying an increasing width Γ. We interpret the turning points as level repulsions of the collective state with the other levels. The descending slope of the envelope of the turning points and the crossing points is due to the finite size of the spectrum.

In Fig. 2 we display the EPs in the Λ plane for a few values of N. The sets of the EPs belonging to the same N are connected by a solid line. The zooming in of the EPs towards
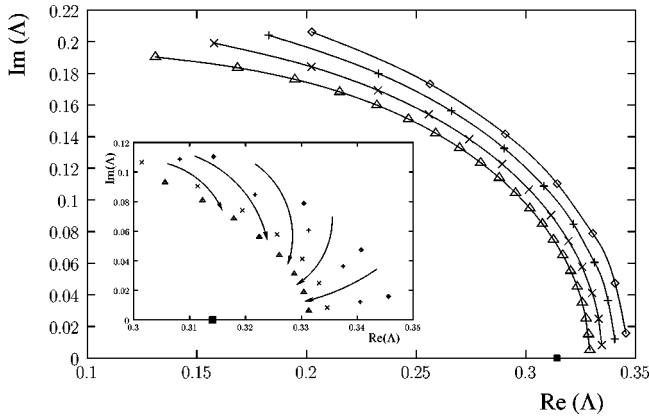
FIG. 2. The exceptional points in the complex Λ plane for $N = 15$ (rhombs), $N = 19$ (plus signs), $N = 27$ (crosses), and $N = 43$ (triangles). The inset is a magnification around the accumulation point Λ_c (black square). The arrows indicate the changes of the EPs with increasing N.
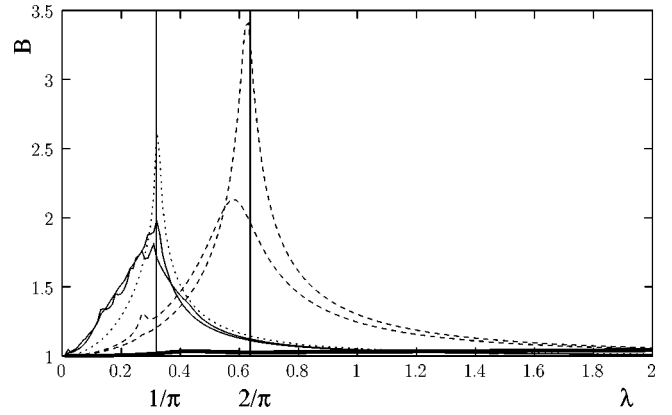

FIG. 3. B as a function of λ for different systems with $\varphi = 0$: ideal picket fence with $N = 101$ (dotted line), randomly perturbed picket fence with $N = 101, 1001$ (solid lines), compensated case with $r = 1$, $t = 4$, and $N = 101, 1001$ (dashed lines), undercompensated case with $r = 0$, $t = 4$, and $N = 101$ (thick line). The two values $\lambda_c = 1/\pi$ and $2/\pi$, referring to $N \to \infty$, are indicated by a vertical solid line.

the accumulation point $\lambda_c = 1/\pi$ for increasing N is clearly discernible and enhanced in the inset of Fig. 2. The larger N the larger is the density of the EPs near to the real axis. Additional points come in further away with each additional step of N. They are typical edge effects and correspond to the turning points of the energy trajectories at the outer edges of the spectrum. These points which emerge further away from the real axis quickly move in to get near to the others while new further points come up for the following steps of N. Eventually, in the limit $N \to \infty$, all points coalesce at the accumulation point.

In this way, variation of the coupling strength Λ invokes a certain trajectory of the system in the complex Λ plane. If it hits the accumulation point Λ_c (or the high-density regions of EPs at finite N), all complex eigenvalues coalesce (nearly coalesce) at their crossing point. This leads to a sharp transition from a regime with N resonance states to a regime with one collective state and $N - 1$ trapped states.

We note that at the accumulation point $\Lambda = \Lambda_c$, which can occur only in the limit $N \to \infty$, all infinitely many eigenstates collapse into one. It is the global collective state. It retains its characteristics also for $\lambda > \lambda_c$, when the other states (the trapped states) re-emerge. This particular case underlines the crucial connection between the critical point of a phase transition and the EPs.

In the picket-fence model the accumulation point Λ_c lies on the real axis. If for physical reasons, a value $\varphi \neq 0$ has to be chosen with H_0 and H_1 left unchanged, a variation of λ will effect a trajectory in the complex Λ plane that passes the high-density region (or accumulation point for infinite N) at a certain distance. This affects the sharpness of the transition between the two regimes. Note that this type of softening of the phase transition is different in nature from the one caused by a finite value of N. It persists in the limit $N \to \infty$. This underlines that φ as well as the distribution of the EPs determine whether there is a phase transition in the strict thermodynamical sense, which is associated with the sudden formation of a globally collective state in the system.

As an aside we realize that the schematic model ($\varphi = 90°$) cannot have any signatures of a sharp phase transition, when the real value of λ is varied from zero to large

values. The accumulation point is then relatively far away on the imaginary axis of the coupling strength. Results for a variety of angles φ but the same H_0 and H_1 have been investigated in [7].

Previously, a criterion for a phase transition, viz.,

$$B = \frac{1}{2N+1} \sum_{k=-N}^{N} \langle \psi_k | \psi_k \rangle, \tag{17}$$

has been introduced in [7]. Excluding the nongeneric case of an accidental crossing of two (or more) eigenvalues at an EP, B is an indicator of the sharpness of the transition. On the one hand, it is the larger the smaller the distance between the complex eigenvalues and their crossing points. This distance is determined by the angle φ. On the other hand, $B \gg 1$ only if all complex eigenvalues reach their minimum distance simultaneously at the same value $\Lambda = \Lambda_c$, i.e., if the EPs accumulate. In this case, B shows a pronounced maximum as a function of Λ around $\Lambda = \Lambda_c$.

In Fig. 3 a few cases including the picket-fence model are illustrated. It also demonstrates a situation where a local collective state is formed, which means that the phase transition is washed out completely. We return to this latter aspect at the end of Sec. III C.

To summarize the findings of this subsection: the high density of EPs provides the mathematical mechanism for the restructuring of the system under variation of the modulus λ of the coupling parameter Λ towards larger values. The sharpness of the transition which is evoked by the variation is determined by the distance at which the high-density region (accumulation point for infinite N) of EPs is passed by the corresponding energy trajectories. This distance is determined by the angle φ.

B. Random change of the unperturbed spectrum

The essential aspects of our findings remain unchanged if the unperturbed energies in H_0 and/or the elements of the coupling matrix V deviate from the symmetrical form used in
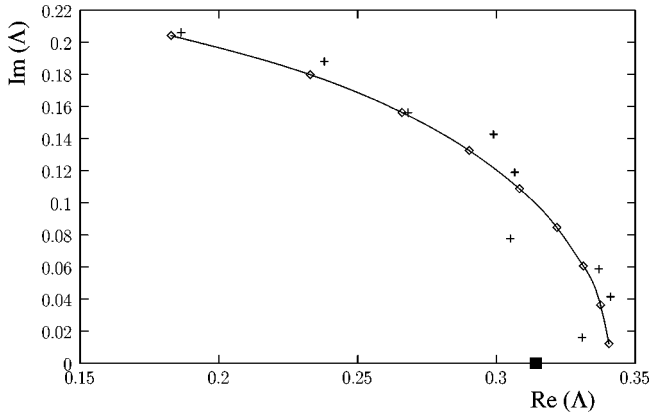
FIG. 4. The EPs for $N=19$ for the ideal picket fence (rhombs) and the randomly perturbed picket fence (plus signs). Λ_c is denoted by a black square.

the preceding section. We address the question of whether and to what extent such disturbance can change the basic pattern, that is, the formation of a region of high density of EPs or even of an accumulation point in the large N limit.

For this purpose, we define random perturbations in H_0 by $(H_0)_{k,k} = -(N-1-2k)/2 + r_k$ where the r_k are random numbers from a uniform distribution in the interval $[-0.1, 0.1]$. In Fig. 4 we have drawn the EPs for the perturbed and unperturbed picket fence in the complex Λ plane for $N=19$. As in Fig. 2, the unperturbed EPs are connected by a solid line. The perturbed EPs no longer lie on the smooth curve but are scattered around it.

The quantity B of Eq. (17) indicates the accumulative behavior of the EPs. It is illustrated in Fig. 3 for the disturbed picket-fence model as a function of λ for two different values of N using $\varphi = 0°$. The excess beyond unity of B around $\lambda = 1/\pi$ increases with increasing N. We interpret this result as a strong indication for the disturbances to be washed out, the more so the larger N. In other words, for an increasing number of states, the average distribution of the EPs is zooming in to $\Lambda_c = \lambda_c = 1/\pi$. Hence, the EPs of the irregular system accumulate on the average.

Furthermore, the precise form of the distribution from which the random changes in H_0 and/or H_1 are drawn is immaterial. In particular, choosing the eigenvalues of H_0 from a Wigner [Gaussian orthogonal ensemble (GOE)] or a Poissonian distribution does not alter our conclusions. The existence and the position of the accumulation point remain unaffected.

C. Level density dependence

In real physical systems, the level density, the number, and the coupling strength of the decay channels are in general energy dependent. The former is usually a monotonically increasing function, whereas, for instance, in nuclear physics the continuum coupling strength decreases. This energy dependence will influence the distribution of the EPs.

In [7] it has been shown that it needs a proper tuning between the density dependence on the one hand and the coupling dependence on the other in order to guarantee a phase transition at a finite value of the coupling strength. Given the energy dependence of the level density $\rho(E)$, a

phase transition occurs at finite coupling strength, if and only if the energy dependence of the coupling vector is given by the inverse function $\rho(E)^{-1}$. If the system obeys this condition on the average, a phase transition still occurs. The critical point may be shifted with respect to the value of the ideal picket-fence model. We talk about ''overcompensation'' when the energy dependence of the coupling vector decreases (increases) at a lesser (faster) rate than that of the inverse behavior of the level density. In this case, numerical results [7] led to the conjecture that the critical point is shifted to zero, i.e., $\lambda_c \to 0$ if $N \to \infty$. In the opposite case, which we denote as ''undercompensation,'' a global collective mode and a global reorganization of the spectrum as a whole is absent. Now one obtains $\lambda_c \to \infty$ if $N \to \infty$. The occurrence of the broad mode remains then a local phenomenon for all finite values of λ [7]. A collapse of the Hilbert space as in the picket-fence model does not occur.

The following substantiates the conjecture about the position of λ_c in the undercompensated and overcompensated cases. To facilitate the discussion we restrict ourselves to a power behavior of the energy dependence of the coupling and level density. The energy dependence of the coupling matrix elements $|v_k|^2$ is assumed to be of the form $|k|^r$. The unperturbed energies (eigenvalues of H_0) are defined as $\epsilon_k = \text{sgn}(k)|k|^{t/2}$. The secular equation Eq. (1) can be written as

$$\sum_{k=-N/2}^{N/2} \frac{|v_k|^2}{\mathcal{E} - \epsilon_k} = \frac{i}{\Lambda}. \tag{18}$$

Using Eq. (18), we obtain

$$\sum_{k=-N/2}^{N/2} \frac{|v_k|^2}{(\mathcal{E} - \epsilon_k)^2} = 0, \tag{19}$$

which corresponds to Eq. (2).

The EPs are simultaneous solutions of Eqs. (18) and (19). It is obvious from Eq. (19) that the energy values at the EP cannot be real. In fact, it is known [7], and also discussed in connection with Fig. 1, that, for $t=2$ and $r=0$ (picket fence), the imaginary parts of the energies at the EP tend to infinity for $N \to \infty$. Numerical evidence as well as the following consideration support the conjecture that this holds for arbitrary values of r and t as long as $t - r > 1$. For this purpose we rewrite Eq. (18) for even N,

$$2\mathcal{E} \sum_{k=1}^{N/2} \frac{k^r}{\mathcal{E}^2 - k^t} = \frac{i}{\Lambda}. \tag{20}$$

In the large N limit we replace the sum in Eq. (20) by an integral and $\mathcal{E}$ by iz and obtain

$$-2iz \int_1^\infty \frac{k^r dk}{z^2 + k^t} = \frac{-2iz}{t-r-1} F_{21}\left(1, 1 - \frac{1+r}{t}, 2 - \frac{1+r}{t}; -z^2\right)$$

$$= \frac{i}{\Lambda}. \tag{21}$$

The derivative with respect to z of the right hand side of Eq. (21), which is a linear combination of two hypergeometric functions, corresponds to the large N limit of Eq. (19) ($\mathcal{E}$

$\rightarrow iz$). It is straightforward to show, for instance, by graphical means, that this derivative has no zero for finite values of z. In fact, the derivative vanishes only for $z \rightarrow \infty$ irrespective of r and t $(t > r + 1)$. We exploit this fact in Eq. (18) in that we consider the limit of the left hand side for large imaginary values of $\mathcal{E} = E - i\Gamma/2$. It is found to be proportional to $\Gamma^{2(1+r)/t}/\Gamma$. This yields the generalized result that the over-compensated case occurs for $2(1+r) > t$, and the compensated and undercompensated cases for $2(1+r) = t$ and $2(1+r) < t$, respectively [7]. In fact, the respective limit of the left hand side of Eq. (21) is infinity, a finite constant, and zero for $\Gamma \rightarrow \infty$. Consequently, the corresponding values of λ must be zero, finite, and infinity, respectively. Since these values constitute simultaneous solutions of Eqs. (18) and (19), they are the values where the EPs accumulate. Furthermore it follows that the accumulation point is on the real axis for the compensated case $[t = 2(1+r)]$.

To demonstrate further this result, the values of B as a function of $\Lambda = \lambda$ ($\varphi = 0$) are drawn for the compensated case $r = 1$, $t = 4$ in Fig. 3. One clearly sees that the maximum of B around $\lambda = 2/\pi$ increases with N. As explained above, this means that, on the average, the minimum distance between the crossing point and its corresponding eigenvalues decreases with increasing N. In the limit $N \rightarrow \infty$, all eigenvalues will hit the crossing point. Thus the compensated system behaves similarly to the picket fence. Also in this case, the space of the eigenstates collapses at $\Lambda = \Lambda_c = 2/\pi$ into dimension one, and a globally collective state is created. For comparison, the undercompensated case with $r = 0$ and $t = 4$ is drawn in Fig. 3 as an example for a system which does not undergo a phase transition. As explained above, $\lambda_c \rightarrow \infty$ in this case and B has no maximum at a finite value of λ.

D. Phase behavior of the wave functions

In Sec. II some properties of the eigenfunctions have been pointed out, when a non-self-adjoint Hamiltonian is considered. We resume the discussion in more detail here with particular emphasis on the phase of the wave functions when an EP is approached by the variation of the coupling parameter Λ. Surrounding one crossing point in the complex energy plane corresponds to a double loop around the EP in the Λ plane, Eq. (7), since the EPs are square root branch points, Eq. (8). We address in the following the effect of looping once around an EP in the Λ plane. For demonstration we use the simple two-dimensional model introduced in Sec. II, Eq. (6).

The two eigenfunctions, normalized according to Eq. (3), can be parametrized by the complex angle θ:

$$\psi_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad \psi_2 = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}, \quad (22)$$

where the angle θ is given by

$$\tan^2\theta = \frac{\mathcal{E}_1 - \mathcal{E}_2 - (\epsilon_1 - \epsilon_2) + i\Lambda \cos 2\omega}{\mathcal{E}_1 - \mathcal{E}_2 + (\epsilon_1 - \epsilon_2) - i\Lambda \cos 2\omega}. \quad (23)$$

The notation has been introduced in Eqs. (6) and (8). From this expression we read off (i) at an EP ($\mathcal{E}_1 = \mathcal{E}_2$) we obtain $\tan^2\theta = -1$ which implies $|\cos\theta| = |\sin\theta| = \infty$, that is, the

components of the wave functions blow up; (ii) when an EP is surrounded in the Λ plane (which amounts to $\mathcal{E}_1 - \mathcal{E}_2 \rightarrow \mathcal{E}_2 - \mathcal{E}_1$), the $\tan^2\theta$ is changed into $1/\tan^2\theta$, which corresponds to the change $\theta \rightarrow \theta + \pi/2$. This implies $\psi_1 \rightarrow \psi_2$ and $\psi_2 \rightarrow -\psi_1$.

While it may not be obvious to implement such a contour in the complex Λ plane in an actual physical experiment, there could be a possibility to achieve the same effect by a variation of the modulus of Λ for different values of the relative coupling given by ω. In fact, using the setting $H_0 - i\lambda V V^\dagger$, then the EP lies for $\omega \gtrsim 45°$ just below and for $\omega \lesssim 45°$ just above the point $\lambda = \epsilon_2 - \epsilon_1$. If we compare the two situations while varying λ, the two wave functions for $\lambda > \epsilon_2 - \epsilon_1$ differ in the same way as if the EP had been surrounded.

This can be made explicit by choosing an expression for $\tan\theta$ which is more convenient for this purpose, viz.,

$$\tan\theta = \frac{-i\lambda \sin 2\omega}{\mathcal{E}_1 - \mathcal{E}_2 + \epsilon_1 - \epsilon_2 - i\lambda \cos 2\omega}. \quad (24)$$

The difference between the two values for ω manifests itself in the difference of the sign of the imaginary part of $\mathcal{E}_1 - \mathcal{E}_2$, since different Riemann sheets have been approached. As a consequence, for large values of λ the right hand side of Eq. (24) tends towards $\tan\omega$ for the one case and towards $-\cot\omega = \tan(\pi/2 + \omega)$ for the other.

We suggest that using electromagnetic resonators may allow control of both the global coupling λ and the relative coupling between two resonances which is given by ω. If it should be possible to bring to interference the wave functions for the two different situations, that is, for $\omega \gtrsim 45°$ and for $\omega \lesssim 45°$, the different phases are expected to be observable.

The situation described is reminiscent of Berry's phase [12]. However, we stress that in our discussion we are dealing with non-self-adjoint operators, and an EP is therefore not to be confused with Berry's diabolic points. This difference was pointed out in [13]. In fact, a generic EP (i.e., when just two levels coalesce) gives rise to a double pole in the corresponding Green's function or S matrix. The mechanism for this to occur is again related to the vanishing of the norm of the eigenvectors when $\Lambda \rightarrow \Lambda_{EP}$. The double pole is an additional signature for an EP. It is different from the simple pole emerging in the Green's function (S matrix), if a usual single resonance or an incidental degeneracy of two resonances occurs, say, owing to some symmetry. The latter situation is, however, an unlikely event as it needs the tuning of four parameters (six parameters in the case of no time reversal) to achieve it. In contrast, the existence of the EPs is an intrinsic mathematical feature.

IV. SUMMARY AND OUTLOOK

In this paper we studied the relation between phase transitions, collective states, and the distribution of EPs of open quantum systems. Since both the collectivity and the phase transition are well pronounced in the one-channel case, we restricted ourselves to an N-dimensional quantum system coupled to one open decay channel. Our results are as follows.

(i) A necessary condition for a phase transition to occur is that the EPs accumulate in the complex Λ plane. This result has been shown analytically for the picket-fence model and some generalizations, and is confirmed by numerical results using the quantity B. The quantity B is a measure for the average minimum distance of the eigenvalues to its crossing point in the complex energy plane. In the cases considered, B has a pronounced maximum at the critical value Λ_c at which the phase transition occurs. The increase of B with the number N of states is a further indication that the EPs accumulate in the complex Λ plane.

(ii) The choice of the angle φ is dictated by the physical situation. Only if it is equal to the phase of Λ_c, the system hits the accumulation point (or goes through the region of high density of EPs), when λ is varied. Otherwise, the system may pass the vicinity of the accumulation point (or the region of high density of EPs) and a phase transition in the strict thermodynamical sense does not occur. In all cases considered, we find that φ has to be $0°$ for hitting the accumulation point Λ_c and for a genuine phase transition to emerge.

(iii) If the system hits the accumulation point, the infinite-dimensional space collapses to a one-dimensional function space, since the infinitely many eigenfunctions are identical at $\Lambda = \Lambda_c$. This is in contrast to the function space of a usual N-fold degeneracy, where N independent eigenfunctions occur. The eigenfunction of the collective state contains contri-butions of all basis functions of the unperturbed system H_0. It is a globally collective state which is created by the system as a whole.

(iv) If the system does not hit the accumulation point, the formation of the collective state is not related to a genuine phase transition. This is, in particular, the case for a system with a purely Hermitian coupling matrix H_1 and $\varphi = 90°$ which creates a state with large internal collectivity.

(v) If the accumulation point is at $\Lambda_c \to \infty$, the formation of the collective state occurs successively and locally. Its wave function contains contributions only from those resonance states which are overlapped by it.

Exploiting the fact that the crossing points in the complex energy plane are square root branch points, we suggest a possibility for an experimental study of the level repulsion of the complex eigenvalues. For this, the relative coupling strength of the states to the continuum as well as the overall value of their coupling strength should be controllable. A signature of local resonance crossing is given by a particular change of the phase of the complex eigenfunctions. Work towards this aim is in progress.

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